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On the low-Rossby-number flow of a rotating fluid past a circular cylinder

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The flow past a cylinder in a rapidly rotating frame is described when the Rossby number Ro is $O(E^{\frac{1}{2}})$, where E is the Ekman number. Previous studies of the configuration have noticed the development of a singularity within the $E^{\frac{1}{4}}$ layer at the rear stagnation point once the ratio $Ro/E^{\frac{1}{2}}$ is larger than a critical value, and concluded that the boundary-layer flow is unsteady. In this paper a description of a steady boundary-layer flow for this parameter range is presented, showing the development of flow separation as $Ro/E^{\frac{1}{2}}$ approaches a larger critical value. Details of the flow once the $E^{\frac{1}{4}}$ layer has separated from the cylinder are also described.

1. Introduction

In this paper the flow of a rotating fluid of finite depth past an axial circular cylinder will be examined when the Ekman number E is small and the Rossby number Ro is $O(E^{\frac{1}{2}})$. Under these conditions Walker & Stewartson (1972) demonstrated that the $E^{\frac{1}{4}}$ layer on the surface of the cylinder is governed by a nonlinear equation and that the flow can separate from the obstacle when the ratio $Ro/E^{\frac{1}{2}}$ is sufficiently large. The phenomenon of $E^{\frac{1}{4}}$ layer separation has also been studied in another configuration by Page (1982), but in the current case the development of separation is complicated by the presence of a singularity at the rear stagnation point of the cylinder. The presence of this singularity, which appears at a smaller value of $Ro/E^{\frac{1}{2}}$ than that beyond which the flow separates, was noted by Walker & Stewartson (1972), who appealed to earlier work on the magnetohydrodynamic flow near a rear stagnation point (Leibovich 1967*a*; Buckmaster 1969, 1971). These studies concluded that once the singularity forms the boundary layer splits into two distinct regions, an inner viscous layer and an outer inviscid layer, and that flow in the outer layer is necessarily unsteady. This is in some disagreement with the careful experiments of Boyer & Davies (1982), who did not notice any unsteadiness in this parameter regime.

The flow past a circular cylinder when $Ro \ll E^{\frac{1}{2}}$ was first described by Barcilon (1970), who calculated the flow in the Stewartson $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers on the surface of the cylinder, appropriate as $Ro/E^{\frac{1}{2}} \to 0$ (Stewartson 1957), and examined some weakly nonlinear effects when $Ro/E^{\frac{1}{2}} \ll 1$. In particular, Barcilon noted that nonlinear effects in the $E^{\frac{1}{3}}$ layers are not significant when $Ro = O(E^{\frac{1}{2}})$, and therefore they will not be considered any further in this paper. Walker & Stewartson (1972) considered the nonlinear effects in the $E^{\frac{1}{4}}$ layer when $Ro = O(E^{\frac{1}{2}})$ in more detail and derived the governing equations, which show some similarity to the boundary-layer equations for a non-rotating fluid. These authors also recognized that the $E^{\frac{1}{4}}$ layer equation was essentially the same as that for the boundary-layer flow in an electrically conducting fluid with a strong radial magnetic field applied, as described by Leibovich (1967*a*) and Buckmaster (1969, 1971). From this work it was apparent that once the ratio

$Ro/E^{\frac{1}{2}}$ exceeded a critical value the flow developed a singularity at the rear stagnation point of the cylinder. For values of $Ro/E^{\frac{1}{2}}$ smaller than this critical value the boundary-layer flow was fully attached and could be described by a similarity solution near the rear stagnation point, but once the critical value was exceeded there was no suitable solution to the similarity equation. Using a technique drawn from classical boundary-layer theory, Buckmaster (1969) showed that the skin friction could not vanish on the surface of the cylinder until $Ro/E^{\frac{1}{2}}$ exceeded a larger critical value, twice that at which the singularity formed, and therefore boundary-layer separation in the classical sense did not necessarily accompany the singularity. Details of the attached flow between these two critical values were examined by Leibovich (1967*a*), who concluded that the boundary-layer flow split into two layers. In the inner 'viscous' layer a similarity solution for the flow could be found, but the velocity at the outer edge of this region did not match that of the exterior flow (outside of the boundary layer). To join these two regions together, Leibovich proposed that an outer 'inviscid' layer was present; however, since he could not find an appropriate description for this flow he concluded that it was necessarily unsteady. Buckmaster (1971) reexamined this layer using a different form of solution, guided by numerical results, but reached a similar conclusion. In this paper a suitable description for the steady flow in this outer layer is proposed and numerical solutions are presented to support this theory.

The results described in §4 show how the boundary layer develops once the rear stagnation-point singularity has formed, and indicate that the transition from attached flow to separated flow is relatively smooth as $Ro/E^{\frac{1}{2}}$ is increased. The main features of the flow in this regime are similar to those proposed by Leibovich (1967*a*), and his description of the flow as undergoing 'separation without reversed flow' seems to be particularly appropriate. Between the inner viscous layer and the outer layer is a region of effectively irrotational stagnation-point flow, contained within the boundary layer, and, as $Ro/E^{\frac{1}{2}}$ is increased, the size of this region also increases. Beyond Buckmaster's (1969) critical value this region forms into a stagnant pool of fluid, surrounded by separated free shear layers, which distort the exterior flow. This description is in broad agreement with the experimental results presented in Boyer (1970) and Boyer & Davies (1982), although the stagnant region will contain some recirculation for finite values of E, and the free shear layers will develop instabilities in practice.

In §2 the governing equations for the $E^{\frac{1}{4}}$ layer flow are derived, and these are examined in §3 for the parameter range where the flow is fully attached and regular at the rear stagnation point. The more interesting case, where the rear stagnation point has developed a singularity but where the flow is still attached, is described in §4. Once the $E^{\frac{1}{4}}$ layer separates from the cylinder, and the exterior flow is distorted, the flow is more difficult to describe, but some proposals on the flow of this parameter regime are outlined in §5.

2. Formulation

Consider an incompressible viscous fluid, of density ρ^* and kinematic viscosity ν^*, which is confined between two infinite parallel plates, a distance d^* apart, and where the entire configuration is rotating with a uniform angular velocity $\Omega^*\hat{\mathbf{k}}$ about an axis perpendicular to the plates. Relative to this rotating system, a circular cylinder of radius l^* is placed in the fluid, with its axis parallel to $\hat{\mathbf{k}}$, and the fluid is forced past the cylinder by imposing a uniform flow with speed U^* at infinity. Based on these dimensional quantities, three important dimensionless parameters can be defined, namely

$$Ro = \frac{U^*}{\Omega^* d^*}, \quad E = \frac{\nu^*}{\Omega^* d^{*2}}, \quad l = \frac{l^*}{d^*}, \tag{2.1}$$

and these will be referred to as the Rossby number, Ekman number and scaled radius respectively. In this paper both Ro and E are considered to be small, and particular emphasis is given to the case where Ro is $O(E^{\frac{1}{2}})$. The scaled radius l is assumed to be of order unity.

The most convenient coordinates to use in this configuration are cylindrical polars where the z^* axis is coincident with the axis of the cylinder and the θ axis is aligned with the imposed velocity at infinity. Dimensionless position and velocity, relative to the rotating frame, can then be defined as $\boldsymbol{x} = (r, \theta, z) = \boldsymbol{x}^*/d^*$ and $\boldsymbol{u} = (u, v, w) = \boldsymbol{u}^*/U^*$ respectively, where d^* and U^* have been chosen as appropriate length and velocity scales. The fluid is therefore contained in the region $r > l$ with $0 < z < 1$, and the velocity tends to $\boldsymbol{u} = (\cos\theta, -\sin\theta, 0)$ as $r \to \infty$. The equations of motion for a steady flow in these coordinates are

$$Ro(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + 2(\boldsymbol{\hat{k}} \times \boldsymbol{u}) = -\boldsymbol{\nabla}P + E\nabla^2\boldsymbol{u}, \tag{2.2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{2.3}$$

where the dimensional pressure p^* has been scaled to the reduced pressure

$$P = \frac{p^* - \frac{1}{2}\rho^*\Omega^{*2}r^{*2}}{\rho^* U^* \Omega^* d^*}, \tag{2.4}$$

after removing the centrifugal contribution (which does not affect the motion in a closed container). The boundary condition $\boldsymbol{u} = \boldsymbol{0}$ is applied on all solid surfaces, namely $r = l$ and $z = 0, 1$.

For $Ro \ll 1$ and $E \ll 1$ the momentum equation (2.2), to lowest order, is the geostrophic equation

$$2(\boldsymbol{\hat{k}} \times \boldsymbol{u}) = -\boldsymbol{\nabla}P, \tag{2.5}$$

and thus the motion in regions where (2.5) is a good approximation is both depth-independent and two-dimensional. Therefore the velocities u, v are functions of (r, θ) only, and, using (2.3), w is zero to lowest order. To evaluate the velocities (u, v) it is necessary to examine higher-order terms in (2.2); this can be achieved most easily by eliminating P from the (r, θ)-components of that equation, leading to

$$Ro\left(u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right) = (2 + Ro\,\zeta)\frac{\partial w}{\partial z} + E\nabla_{\mathrm{h}}^2\zeta, \tag{2.6}$$

which governs the z-component of vorticity,

$$\zeta = \frac{1}{r}\frac{\partial(rv)}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}, \tag{2.7}$$

to lowest order. In (2.6) the symbol ∇_{h}^2 represents the two-dimensional Laplacian, which arises because the vorticity ζ in (2.7) is independent of z. It follows that the term $\partial w/\partial z$ in (2.6) is a function of (r, θ) only and hence that w is linear in z throughout the geostrophic region. By examining the ageostrophic Ekman layers on $z = 0, 1$ (Greenspan 1968) it can be shown that w is $O(E^{\frac{1}{2}})$ and that

$$\frac{\partial w}{\partial z} = -E^{\frac{1}{2}}\zeta \tag{2.8}$$

outside of these layers. Neglecting the term $Ro\,\zeta$ in (2.6), which can be shown to be small when $Ro \ll E^{\frac{1}{3}}$ (Page 1983), and defining the parameters

$$\lambda = \frac{Ro}{2E^{\frac{1}{2}}}, \quad \delta = (\tfrac{1}{2}E^{\frac{1}{2}})^{\frac{1}{2}}, \tag{2.9}$$

the vorticity equation becomes

$$\lambda\left(u\frac{\partial \zeta}{\partial r}+\frac{v}{r}\frac{\partial \zeta}{\partial \theta}\right) = -\zeta+\delta^2\nabla_{\mathrm{h}}^2\zeta. \tag{2.10}$$

This equation, together with the continuity equation

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta} = 0 \tag{2.11}$$

and the boundary conditions on (u, v), is sufficient to determine the velocities everywhere in the geostrophic region, which includes the $E^{\frac{1}{4}}$ layer on the cylindrical surface.

It is convenient to introduce a stream function $\psi(r,\theta)$ defined by

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v = \frac{\partial \psi}{\partial r}, \tag{2.12}$$

so that (2.10) becomes

$$\lambda \mathrm{J}(\psi,\zeta) = -\zeta+\delta^2\nabla_{\mathrm{h}}^2\zeta, \tag{2.13}$$

where J is the Jacobian in radial polars and $\zeta = \nabla_{\mathrm{h}}^2\psi$. The boundary conditions to be satisfied by ψ are

$$\psi(l,\theta) = \frac{\partial \psi}{\partial r}(l,\theta) = 0 \tag{2.14}$$

and $\psi \rightarrow -r\sin\theta$ as $r \rightarrow \infty$.

When $\delta \ll 1$ it is clear that the diffusive term $\delta^2\nabla_{\mathrm{h}}^2\zeta$ in (2.10) can be neglected everywhere outside of a thin boundary layer, of thickness $O(\delta)$, which is primarily against the surface of the cylinder. The vorticity equation can then be written in the form

$$\lambda|\boldsymbol{u}|\frac{\partial \zeta}{\partial s} = -\zeta, \tag{2.15}$$

where $|\boldsymbol{u}|$ is the speed of the flow and s is the distance along a streamline. It follows that ζ decays exponentially along streamlines and, since $\zeta \rightarrow 0$ as $r \rightarrow \infty$, that $\zeta = 0$ along streamlines originating from infinity. It can also be shown that $\zeta = 0$ along closed streamlines, which may form in the wake of the cylinder.

Assuming, initially, that the boundary layer is attached to the cylinder, then the appropriate stream function for the flow in the exterior, outside of the $E^{\frac{1}{4}}$ layers, is

$$\psi_{\mathrm{e}} = -\left(r-\frac{l^2}{r}\right)\sin\theta, \tag{2.16}$$

where the subscript identifies the exterior flow. The resultant tangential velocity against the surface of the cylinder can be calculated from (2.16) to be

$$\frac{\partial \psi_{\mathrm{e}}}{\partial r}(l,\theta) = -2\sin\theta, \tag{2.17}$$

so that clearly the second condition in (2.14) is not satisfied by the exterior flow. This velocity discontinuity is accommodated by a boundary layer of scale thickness $E^{\frac{1}{4}}$ (Barcilon 1970; Walker & Stewartson 1972), known as an $E^{\frac{1}{4}}$ layer, in which the diffusive term in (2.13) becomes important.

To resolve the $E^{\frac{1}{4}}$ layer a new set of scaled coordinates can be introduced, defined by

$$\bar{r} = \frac{r-l}{\delta}, \quad s = \pi-\theta, \qquad \bar{u} = \frac{ul}{2\delta}, \quad \bar{v} = -\frac{v}{2}, \tag{2.18}$$

where attention from this point is to be restricted to the upper half of the cylinder with $0 \leqslant s \leqslant \pi$ (or $0 \leqslant \theta \leqslant \pi$). It might be noted that asymmetric effects for this configuration are discussed by Merkine & Solan (1979), but in the formal limit $\delta \rightarrow 0$ these may be neglected. In terms of these new quantities, the vorticity, to lowest order, is

$$\zeta = -\frac{2}{\delta}\frac{\partial \bar{v}}{\partial \bar{r}}, \tag{2.19}$$

and the vorticity equation (2.13) can be integrated with respect to $\bar{r}$, once higher-order terms have been neglected, to give

$$\frac{2\lambda}{l}\left(\bar{v}\frac{\partial \bar{v}}{\partial s}+\bar{u}\frac{\partial \bar{v}}{\partial \bar{r}}\right) = \frac{2\lambda}{l}\bar{v}_{\mathrm{e}}\frac{\mathrm{d}\bar{v}_{\mathrm{e}}}{\mathrm{d}s}+\bar{v}_{\mathrm{e}}-\bar{v}+\frac{\partial^2 \bar{v}}{\partial \bar{r}^2}, \tag{2.20}$$

In this equation $\bar{v}_{\mathrm{e}}$ is the tangential velocity at the outer edge of the boundary layer, which for fully attached flow is $\bar{v}_{\mathrm{e}} = \sin s$ from (2.17). The momentum equation (2.20) and the continuity equation

$$\frac{\partial \bar{v}}{\partial s}+\frac{\partial \bar{u}}{\partial \bar{r}} = 0 \tag{2.21}$$

are sufficient to determine the flow in the boundary layer, subject to the boundary conditions $\bar{u} = \bar{v} = 0$ on $\bar{r} = 0$, $\bar{v} \rightarrow \bar{v}_{\mathrm{e}}$ as $\bar{r} \rightarrow \infty$, and the initial condition $\bar{v} = 0$ at $s = 0$.

In the particular case where $\lambda = 0$, an exact solution to (2.20) can be obtained, namely

$$\bar{v} = \bar{v}_{\mathrm{e}}[1-\exp(-\bar{r})], \tag{2.22}$$

showing that the boundary layer for $Ro \ll E^{\frac{1}{2}}$ is fully attached with a constant displacement thickness. However, for $\lambda > 0$ the equation (2.20) is parabolic and must be integrated numerically as, for example, in Crissali & Walker (1976). This equation is also equivalent to each of the $E^{\frac{1}{4}}$ layer equations studied by Page (1982, 1983), who concluded that (2.20) is very similar in character to the classical boundary-layer equation in a non-rotating fluid. In particular, for sufficiently large values of λ, boundary-layer separation and wake formation can be expected to occur.

At this point an analogy can be drawn with the magnetohydrodynamic problem of the two-dimensional flow of an electrically conducting viscous fluid past a circular cylinder in the presence of a magnetic field normal to the cylinder wall (Leibovich 1967*a*; Buckmaster 1969). Introducing the parameter $N = l/2\lambda$, similar to that introduced by Walker & Stewartson (1972), (2.20) can be written as

$$\bar{v}\frac{\partial \bar{v}}{\partial s}+\bar{u}\frac{\partial \bar{v}}{\partial \bar{r}} = \bar{v}_{\mathrm{e}}\frac{\mathrm{d}\bar{v}_{\mathrm{e}}}{\mathrm{d}s}+N(\bar{v}_{\mathrm{e}}-\bar{v})+N\frac{\partial^2 \bar{v}}{\partial \bar{r}^2}. \tag{2.23}$$

The parameter N used here corresponds to the parameter $\mathscr{N}$ defined in Leibovich (1967*a*), to N_{L} used in Buckmaster (1969), to $\frac{1}{2}N$ used in Walker & Stewartson (1972, 1974) and Crissali & Walker (1976), and to N used in Buckmaster (1971). The limit

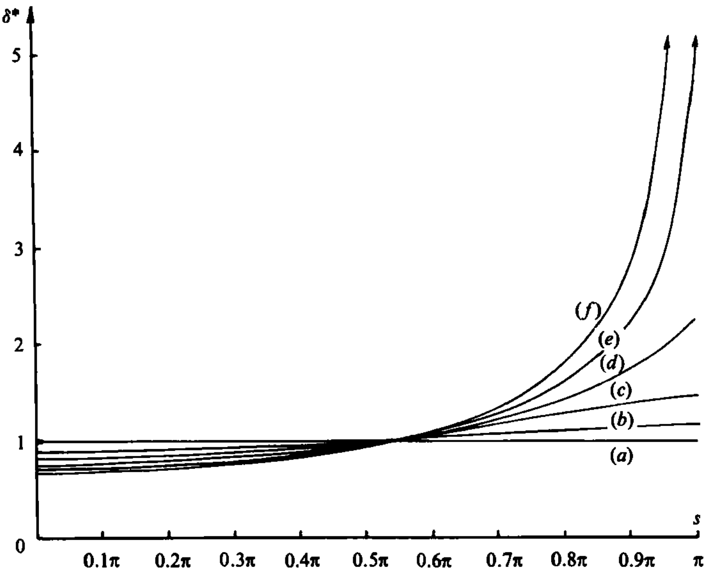

FIGURE 1. Displacement thickness δ^* of the $E^{\frac{1}{4}}$ layer around the cylinder for (*a*) $N = \infty$; (*b*) 10; (*c*) 5; (*d*) $\frac{10}{3}$; (*e*) $\frac{5}{2}$; (*f*) 2.

$N \rightarrow \infty$ corresponds to the limiting case $\lambda = 0$ and the solution (2.22), with nonlinear effects increasing as N is decreased.

In the following sections the flow in the boundary layer on the cylinder will be examined over three important parameter ranges. Since most of the previous work on this problem has been in the magnetohydrodynamic context, the parameter N will be used in these sections rather than λ, which is more popular in the context of a rotating fluid. This will enable the results of this paper to be compared directly with those of Leibovich (1967*a*) and Buckmaster (1971). The parameter λ will be reintroduced in §6, where the results of §§3–5 will be summarized.

3. The case $N \geqslant 2$

For large values of N the leading-order solution to (2.23) is given by (2.20), and therefore the boundary layer on the cylinder induces only a small uniform displacement effect on the exterior flow, effectively increasing the radius of the cylinder by an amount δ. As N is decreased, the inertial effects become more important in the boundary layer, and these tend to reduce the displacement thickness on the upstream side of the cylinder, and increase the displacement thickness on the downstream side, where the flow decelerates. This feature was noted by Barcilon (1970), who used multiple-scale analysis to determine the leading-order correction to the displacement thickness δ^* due to inertial effects. A more general version of Barcilon's result can be derived from equation (2.23) for $N \gg 1$, giving

$$\delta^* = 1 - \tfrac{5}{4}N^{-1}\frac{\mathrm{d}\bar{v}_{\mathrm{e}}}{\mathrm{d}s} + O(N^{-2}), \tag{3.1}$$

so that $\delta^* \approx 1 - \frac{5}{4}N^{-1}\cos s$ in this case. To examine the variation of δ^* around the cylinder for various values of N, the solution to (2.23) with $\bar{v}_{\mathrm{e}} = \sin s$ was calculated numerically, and the results are plotted on figure 1. The numerical method used was based on the 'box method' of Keller & Cebeci (1971) using a scaling in the $\bar{r}$-direction, similar to that described in Crissali & Walker (1976). The plot of δ^* for $N = 10$ is in close agreement with the approximate expression given by (3.1), but as N is decreased further the boundary-layer thickening becomes very significant near $s = \pi$. In fact, the analysis to follow indicates that δ^* becomes unbounded as $s \to \pi$ once $N \leqslant 3$, confirming the presence of a singularity at the rear stagnation point originally recognized by Leibovich (1967*a*). In contrast, the forward stagnation point remains regular for all N, and it can be shown that as $N \to 0$ the flow in this region is similar to the corresponding flow in a non-rotating fluid, but with the scale thickness of the boundary layer decreasing proportionally to $N^{\frac{1}{2}}$.

The most important feature of the boundary-layer flow is the development of the singularity at the rear stagnation point, which becomes apparent in the numerical solutions to (2.23) for $N \leqslant 3$. The flow in the vicinity of this point has been studied by Leibovich (1967*a*), who considered a linear external flow which corresponds to $\bar{v}_e = \pi - s$ in the present coordinates. The features of this flow should be very similar to those for $\bar{v}_e = \sin s$, since $\sin s \sim \pi - s$ as $s \to \pi$. Leibovich (1967*a*) assumed that a similarity solution of (2.23) can be found with $\bar{v} = (\pi - s) f'(\bar{r})$, and showed that f must then satisfy an equation equivalent to

$$Nf''' - ff'' + (f'-1)(f'-N+1) = 0, \tag{3.2}$$

with boundary conditions $f(0) = f'(0) = 0$ and $f'(\infty) = 1$. Leibovich (1967*b*) showed that there exists a unique solution to this problem for $N \geqslant 2$, and therefore the assumed form of the solution appears to be suitable for this parameter range. Leibovich (1967*a*) also noted that no such solution can exist for $N < 2$, although a solution of (3.2) satisfying $f'(\infty) = N - 1$ does exist; this solution will be discussed further in §4.

One feature of the solutions of (3.2) for $2 < N < \infty$ is their algebraic decay at the outer edge of the boundary layer since

$$f' \sim 1 + \mathrm{const}\, \bar{r}^{2-N} \tag{3.3}$$

(Leibovich 1967*a*). Integrating once implies that

$$f \sim \bar{r} + a\bar{r}^{3-N} + b \tag{3.4}$$

where a, b are constants which can, in principle, be determined from the numerical solution of (3.2). When $N = 2$ the second term in (3.4) is of the same magnitude as the leading term, and the asymptotic expansion breaks down. In this case it can be shown that

$$f' \sim 1 - \frac{1}{\ln \bar{r} + c}, \tag{3.5}$$

where c is a constant, so that the velocity tends to the exterior flow very slowly in $\bar{r}$. This indicates that the boundary layer has become very thick in the vicinity of the rear stagnation point, and signals the breakdown of this particular similarity solution for $N < 2$. The displacement thickness $\delta^* = \lim_{\bar{r}\to\infty} (\bar{r} - f)$ can be calculated from the expansion (3.4), and for $N > 3$ it is clear that $\delta^* = -b$. However, once $N \leqslant 3$ the displacement thickness becomes infinite at the rear stagnation point, as was indicated in the numerical results shown in figure 1.

The asymptotic behaviour of f for large values of $\bar{r}$ leads to some difficulties with the matching of the boundary-layer flow into the exterior flow, particularly when $N \leqslant 3$. From (3.3) f'' is proportional to $\bar{r}^{1-N}$, and therefore the vorticity at the outer edge of the boundary layer is proportional to $(\pi - s)(r - l)^{1-N}$ and of order δ^{N-2}. For

$N > 2$ this vorticity advected out of the boundary layer will only slightly perturb the $O(1)$ exterior flow, so the streamlines in this region will be given by (2.16) to leading order. The second-order correction to this flow will have non-zero vorticity and satisfy the equation

$$\frac{l}{r}\left(\frac{\partial \psi_e}{\partial r}\frac{\partial \zeta}{\partial \theta} - \frac{\partial \psi_e}{\partial \theta}\frac{\partial \zeta}{\partial r}\right) = -2N\zeta, \tag{3.6}$$

from (2.15). This equation can be solved in the vicinity of the rear stagnation point by changing coordinates from (r, θ) to (r, ψ_e), so that

$$l \cos\theta\left(1 - \frac{l^2}{r^2}\right)\frac{\partial \zeta}{\partial r} = -2N\zeta, \tag{3.7}$$

and hence

$$\zeta = Z(\psi_e)\left(\frac{r+l}{r-l}\right)^N \exp\left(\frac{-2Nr}{l}\right) \tag{3.8}$$

for $\theta \ll 1$. Here Z is a function of ψ_e which is determined by matching onto the boundary layer at $r = l$. To this end note that $\psi_e \sim \delta(\pi - s)\,\bar{r}$ as $r \to l$, and therefore

$$\zeta \propto Z(\delta(\pi - s)\,\bar{r})\,\bar{r}^{-N} \tag{3.9}$$

at the outer edge of the boundary layer. Since ζ should be zero on $\theta = 0$ (or $\psi_e = 0$), we might expect that Z is approximately linear for small values of its argument, which would then imply that ζ is proportional to $(\pi - s)\,\bar{r}^{1-N}$. Therefore (3.8) matches directly onto the vorticity shed by the boundary layer, through the second term in (3.3), and also indicates that the vorticity decays exponentially in a wake behind the cylinder. This exponential decay, over a distance $O(N^{-1})$, is similar to that behind the flat plate in Page (1983). It is also worth noting that this wake will be confined to the region where $\psi_e = O(\delta)$ since the streamlines outside this region will not have entered the boundary layer. Therefore Z will only be non-zero for ψ_e of order δ, which, for finite r, implies that the wake is of thickness $O(\delta)$ in θ. For $\theta \ll \delta$ the analysis above indicates that

$$\zeta \propto \frac{\theta}{r}\frac{(r+l)^{N+1}}{(r-l)^{N-1}} \exp\left(\frac{-2Nr}{l}\right), \tag{3.10}$$

since Z is linear in ψ_e for $\psi_e \ll \delta$.

For $N = 2$ the vorticity shed by the boundary layer, through the second term in (3.5), is of order $(\ln \delta)^{-1}$ in the wake, but once $N < 2$ it can be expected that the wake will contain vorticity of order unity at least, albeit in a thin layer.

4. The case $1 \leqslant N < 2$

As was noted in §3, there is no suitable solution of (3.2) satisfying $f'(\infty) = 1$ when $N < 2$, although there is a solution that satisfies $f'(\infty) = N - 1$. At first this solution might not seem acceptable, since it would imply that $\bar{v} \sim (N-1)(\pi - s)$ at the outer edge of the boundary layer, and hence would not match onto the exterior flow $\bar{v}_e = \pi - s$. However, Leibovich (1967*a*) proposed that the boundary-layer flow in this parameter regime consists of two parts: an inner viscous layer in which the similarity solution above is valid and an outer inviscid layer which provides a transition between $\bar{v} = (N-1)(\pi - s)$ and $\bar{v}_e$. This hypothesis was supported by the numerical evidence presented in Buckmaster (1971), but when both Leibovich (1967*a*) and Buckmaster (1971) attempted to determine the structure of the outer layer they found inconsistencies in their solutions. As a consequence, both authors concluded that no

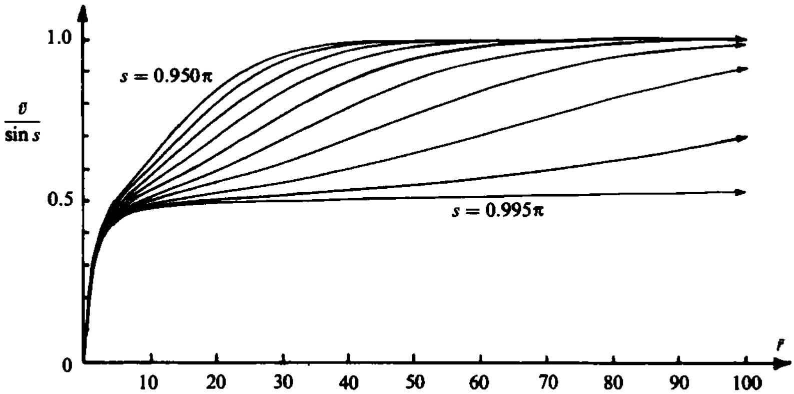

FIGURE 2. Profiles of the scaled $E^{\frac{1}{4}}$ layer velocities $\bar{v}/\bar{v}_e$ close to the rear stagnation point, for $N = \frac{3}{2}$, shown between $s = 0.950\pi$ and 0.995π at intervals of 0.005π.

steady-flow solution could be found for the outer layer. One aim of this paper is to demonstrate that a steady-flow solution can be found, but that it is not of the simple form proposed in the previous studies.

The lower limit on the parameter range examined in this section is imposed by the vanishing of the velocity within the inner layer proposed above, which occurs when $N = 1$. At this same value of N the scaled skin friction

$$\frac{\partial \bar{v}}{\partial \bar{r}}(s,0)\Big/\sin s$$

vanishes at the rear stagnation point (Buckmaster 1971) which suggests that this quantity will be negative for $N < 1$, and hence that separation will have occurred at some $s < \pi$. These features are supported by the numerical solutions in Crissali & Walker (1976). Buckmaster (1971) also derived a necessary condition for the occurrence of separation, based on the classical requirement that $(\partial^2\bar{v}/\partial\bar{r}^2)(s,0) \geqslant 0$ at the point where the skin friction vanishes, which requires that

$$\frac{\mathrm{d}\bar{v}_e}{\mathrm{d}s} + N \leqslant 0. \tag{4.1}$$

This is satisfied at the rear stagnation point for $N \leqslant 1$, so it appears likely that $N < 1$ is both a necessary and sufficient condition for boundary-layer separation. In addition, (4.1) implies that separation, in the usual sense, cannot occur when $1 \leqslant N < 2$.

To reexamine Leibovich's (1967*a*) hypothesis that the boundary-layer flow splits into two layers when $1 \leqslant N < 2$, (2.23) was integrated numerically with $\bar{v}_e = \sin s$ for various values of N. As was noted in Crissali & Walker (1976), there is a rapid thickening of the boundary layer near the rear stagnation point, and consequently a special scaling, to be described later in this section, was used to help resolve this feature. The results of these calculations are presented in figure 2, where $\bar{v}/\sin s$ is plotted as a function of $\bar{r}$ for ten values of s near $s = \pi$. This figure shows the development of the inner layer, with $\bar{v}/\sin s \approx N-1 = \frac{1}{2}$ at its outer edge, and a rapidly thickening outer layer which provides a smooth transition between $\bar{v}/\sin s = N-1$ and $\bar{v}/\sin s = 1$. These numerical solutions also indicate that the displacement thickness of the boundary-layer flow for $N = \frac{3}{2}$ increases slightly faster than $O(\pi-s)^{-1}$ as $s\to\pi$, as was noted by Buckmaster (1971). However, the most important feature of these results is that they present a consistent steady solution to (2.23) for a value of N at which both Leibovich (1967*a*) and Buckmaster (1971) concluded that no steady solution could exist. Before reexamining their solutions for the outer flow, a few important features of the inner layer need to be outlined.

Leibovich (1967*a*) concluded that in the inner 'viscous' layer the similarity solution

$$\bar{v}=(\pi-s)f'(\bar{r}),$$

where f satisfies (3.2), is appropriate and that $\bar{v}\to(\pi-s)(N-1)$ at the outer edge of this layer. In fact, the solutions of (3.2) for $1\leqslant N<2$ are closely related to those for $N>2$; if $f(\bar{r})$ is a solution of (3.2) with $f'(\infty)=N-1$ for $1\leqslant N<2$ and $\tilde{f}(\bar{r})$ is a solution with $\tilde{f}'(\infty)=1$ for $\tilde{N}=N/(N-1)$ then

$$f(\bar{r})=\frac{N}{\tilde{N}}\tilde{f}(\bar{r})=(N-1)\tilde{f}(\bar{r}). \tag{4.2}$$

This one-to-one relationship means that the properties of $f(\bar{r})$ for $1\leqslant N<2$ can be obtained by applying a simple transformation to the properties of $f(\bar{r})$ for $N>2$. In particular, (3.3) implies that

$$f'\sim(N-1)+d\bar{r}^{(N-2)/(N-1)}\quad\text{for } \bar{r}\gg 1, \tag{4.3}$$

where d is a constant, which can then be used to match the inner layer onto the outer 'inviscid' layer. Another important feature of the inner layer is that the stream function ψ, defined by

$$\bar{u}=-\frac{\partial\bar{\psi}}{\partial s},\quad \bar{v}=\frac{\partial\bar{\psi}}{\partial\bar{r}}, \tag{4.4}$$

and related to that defined in (2.12) through $\bar{\psi}=-\psi/2\delta$, is given by $\bar{\psi}=(\pi-s)f(\bar{r})$. As a result $\bar{\psi}$ is $O(\pi-s)$ in this layer as $s\to\pi$, a property that will later be used to identify it in some of the numerical solutions.

Using (4.3) the sizes of each term in (3.2) for $\bar{r}\gg 1$ can be compared and it is apparent that the viscous term Nf''' becomes less important at the outer edge of the inner layer. Therefore, it is reasonable to expect that the flow beyond this region can be described by an inviscid version of equation (2.23), in which the term $N\partial^2\bar{v}/\partial r^2$ is omitted, namely

$$\bar{V}\frac{\partial\bar{V}}{\partial s}+\bar{U}\frac{\partial\bar{V}}{\partial\bar{r}}=\bar{V}_{\mathrm{e}}\frac{\mathrm{d}\bar{V}_{\mathrm{e}}}{\mathrm{d}s}+N(\bar{V}-\bar{V}_{\mathrm{e}}), \tag{4.5}$$

where $\bar{U}$, $\bar{V}$ have been used to identify the outer flow velocities. Here the external flow is assumed to be $\bar{V}_{\mathrm{e}}=\pi-s$, as in §3. Leibovich (1967*a*) sought a solution of (4.5) of the form $\bar{V}=(\pi-s)F_{\mathrm{L}}(\bar{r})$ and correctly surmised that no such solution can provide the necessary transition between $F_{\mathrm{L}}=N-1$ at the inner edge and $F_{\mathrm{L}}=1$ against the external flow. As a result he concluded that the flow in this outer layer must be unsteady, and he presented an analysis based on Proudman & Johnson (1962) in support of this conclusion. Buckmaster (1971) reexamined this problem and, on the basis of numerical evidence, sought a more general solution of (4.5) of the form $\bar{V}=(\pi-s)F_{\mathrm{B}}(\bar{r}(\pi-s)^{\alpha})$ for some constant $\alpha>0$. One disturbing feature of this solution is the indeterminacy of both α and a constant scaling factor for $\bar{r}$; Buckmaster, quite correctly, deduces that these constants would be determined by the upstream flow, but makes no attempt to demonstrate that two free constants are sufficient to match onto the numerical solutions. Apart from this difficulty, the

Buckmaster solution does give qualitative agreement with the numerical results, provided $\alpha > 1$. However, when higher-order terms are examined a contradiction is reached, and Buckmaster ultimately concludes that the flow must be unsteady. It is relevant to note that Boyer & Davies (1982) saw no evidence of unsteady flow in their experiments within this parameter regime; in fact they did not observe unsteady flow until N was significantly smaller than one, and the boundary layers had already separated from the cylinder.

In this study a more general solution to (4.5) is sought using a technique similar to that used in Stewartson & Simpson (1982) and Brown & Simpson (1982), namely to replace the coordinate $\bar{r}$ with a stream-function coordinate $\bar{\psi}$ defined using (4.4). Equation (4.5) in $(s, \bar{\psi})$-coordinates is therefore

$$\bar{V}\frac{\partial \bar{V}}{\partial s} = \bar{V}_e \frac{d\bar{V}_e}{ds} + N(\bar{V}_e - \bar{V}), \tag{4.6}$$

and for $\bar{V}_e = \pi - s$ we seek a solution of the form $\bar{V} = (\pi - s)\, F(s, \bar{\psi})$. Since $\bar{\psi}$ does not appear explicitly in (4.6), the equation can be integrated to give

$$(1-F)\,|\,F+1-N\,|^{1-N}(\pi-s)^{2-N} = \mathscr{F}(\bar{\psi}), \tag{4.7}$$

where $\mathscr{F}$ is an unknown function of $\bar{\psi}$ which can only be determined from the numerical solutions for the flow upstream of the stagnation point. Once $\mathscr{F}$ is known, the function $F(s, \bar{\psi})$, and hence $\bar{V}$, can be calculated from (4.7) for any value of $s < \pi$. The vertical velocity $\bar{U}$ is then determined through the continuity equation, giving

$$\bar{U} = \bar{V}\int_0^{\bar{\psi}} \frac{\partial}{\partial s}\left(\frac{1}{\bar{V}}\right) d\bar{\psi}. \tag{4.8}$$

Buckmaster's (1971) solution of the form $\bar{V} = (\pi - s)\, F_B(\bar{r}(\pi - s)^{\alpha})$ assumes, in effect, that $\mathscr{F}$ is a power of $\bar{\psi}$, namely

$$\mathscr{F}(\bar{\psi}) = \frac{1}{a}\,\bar{\psi}^{(N-2)/(\alpha-1)}, \tag{4.9}$$

where a and α are the undetermined constants used in his paper. The relationship (4.9) is not immediately obvious, but it can be established from his equation (3.5), and therefore the validity of his solution depends on whether the numerical results show $\mathscr{F}$ to be of this simple form.

The numerical solutions of (2.23) for this parameter regime were calculated in a similar manner to those for $N \geqslant 2$, but with one important difference. Previous studies (Buckmaster 1971; Crissali & Walker 1976) have noted that the thickness of the boundary layer increases rapidly as $s \to \pi$ when $1 \leqslant N < 2$, and therefore, to resolve this feature properly, a scaled coordinate was introduced in the $\bar{r}$ direction. Since the form of (4.7) indicates that $\bar{\psi}$ remains finite in the outer layer as $s \to \pi$, this new coordinate should ideally follow the streamlines of the flow. For this reason the coordinate $\bar{R} = \bar{r}\cos\frac{1}{2}s$ was chosen because $\bar{R}$ is roughly proportional to $\bar{\psi}$ in the most of the flow, in particular at the outer edge of the boundary layer near $s = \pi$ where $\bar{\psi} \propto \bar{r}(\pi - s)$. Since $s = \pi$ is a singularity in this transformation, the numerical results were terminated before this point, typically at $s = 0.995\pi$. To ensure that the inner layer, in which $\bar{\psi}$ is $O(\pi - s)$, was resolved properly, a very small $\bar{R}$-spacing was used near the wall.

The relationship (4.7) is strictly only valid for the external flow $\bar{V}_e = (\pi - s)$ so the quantity

$$\mathscr{G} = (\pi - s - \bar{V})\,|\,\bar{V} - (N-1)(\pi - s)\,|^{1-N}, \tag{4.10}$$

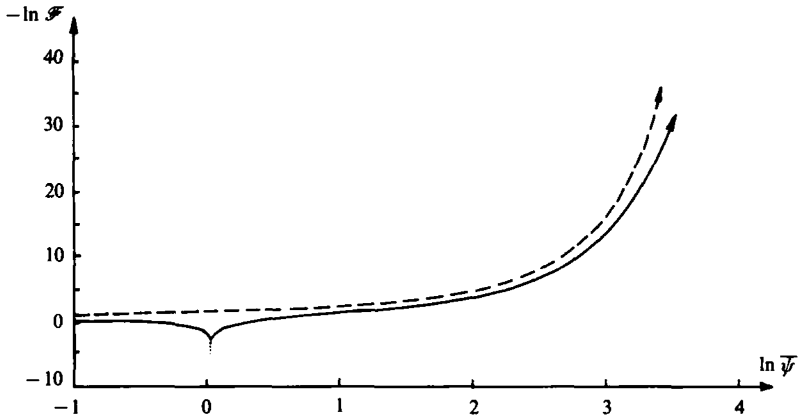

FIGURE 3. Plots of $-\ln \mathscr{F}$, where $\mathscr{F}$ is defined in (4.7), as a function of $\ln \bar{\psi}$ obtained from the numerical solution for (*a*) $N = \frac{5}{3}$ (——); (*b*) $\frac{4}{3}$ (--------).

obtained from the left-hand side of (4.7), will not be independent of s for the flow near the rear stagnation point of the cylinder. However, since $\sin s \sim \pi - s$ near $s = \pi$ the values of $\mathscr{G}$ should tend smoothly to a non-trivial limiting function $\mathscr{F}(\bar{\psi})$ as $s \to \pi$, and this feature is confirmed by the numerical solutions. This limiting function is plotted in figure 3 for two different values of N, using a logarithmic scale on both ordinates to facilitate comparison with (4.9). If that equation were valid then $\ln \mathscr{F}$ would be linear in $\ln \bar{\psi}$, but clearly this is not the case, so Buckmaster's assumed form of solution is not appropriate. Figure 3 also indicates that $\mathscr{F}$ decays rapidly in $\bar{\psi}$ as $\bar{\psi} \to \infty$; in fact plots of $\ln \mathscr{F}$ against $\bar{\psi}$ show that this decay is at least exponential in $\bar{\psi}$, which is much more satisfactory than the algebraic decay implied by (4.9). The apparent singularity on the $N = \frac{5}{3}$ curve near $\bar{\psi} = 1$ does not actually represent any singularity in the flow, but rather it corresponds to the value of $\bar{\psi}$ where $F = N - 1$ and the left-hand side of (4.7) is infinite. A similar singularity is present for $N = \frac{4}{3}$, but it is beyond the range of $\bar{\psi}$ plotted on figure 3.

When $\bar{\psi}$ is $O(\pi - s)$, viscous effects become important in the boundary layer, and (4.5) is no longer accurate, so in this region the inner layer must be matched onto the outer layer given by $\mathscr{F}$. The matching condition between these layers is not $F = N - 1$, as supposed earlier, since figure 3 indicates that this value is attained at a non-zero value of $\bar{\psi}$. In fact, the outer layer extends into the region where $F < N - 1$ and matches onto the second term in (4.3), which implies that

$$f' - N + 1 \sim d\bar{r}^{(2-N)/(1-N)} \tag{4.11}$$

for $\bar{r} \gg 1$. From the solution in the inner layer it follows that $\bar{\psi} = (N-1)(\pi - s)\bar{r}$ in this region, and therefore (4.11) implies that

$$F - N + 1 \sim d\left[\frac{\bar{\psi}}{(N-1)(\pi - s)}\right]^{(2-N)/(1-N)} \tag{4.12}$$

for $\bar{\psi} \ll 1$. Inserting this result into (4.7) implies that $\mathscr{F}$ is proportional to $\bar{\psi}^{2-N}$ as $\bar{\psi} \to 0$, and therefore the matching between the layers depends on whether this is confirmed by the numerical solutions. To examine this, the quantity $(\bar{\psi}^{N-2}\mathscr{G})^{1/(1-N)}$ is plotted in figure 4 for $N = \frac{5}{3}$ and several values of s near $s = \pi$ (the power $1/(1-N)$ is introduced to remove the singularity in $\bar{\psi}^{N-2}\mathscr{G}$ when $F = N - 1$). From this figure it is clear that $\bar{\psi}^{N-2}\mathscr{G}$ is approaching a limiting function as $s \to \pi$ and that this function

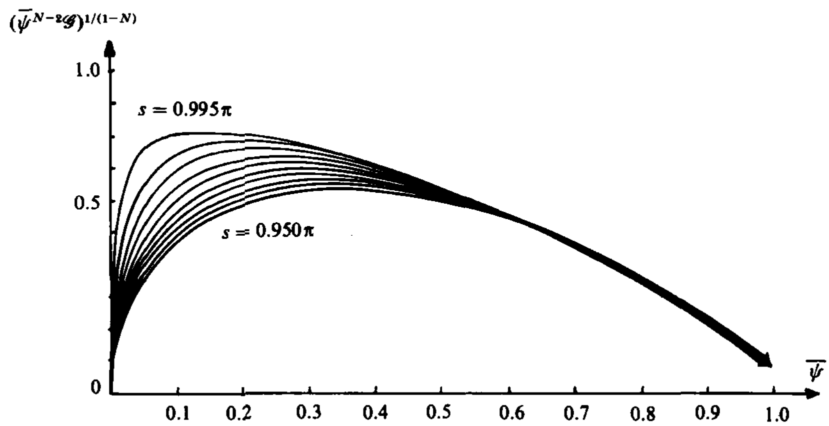

FIGURE 4. Plots of $(\bar{\psi}^{N-2}\mathscr{G})^{1/(1-N)}$ as a function of $\bar{\psi}$ near the inner edge of the boundary layer between $s = 0.950\pi$ and 0.995π at intervals of 0.005π.

is finite and non-zero at $\bar{\psi} = 0$, confirming that $\mathscr{F} \propto \bar{\psi}^{2-N}$ as $\bar{\psi} \to 0$. It is also apparent that the inner region is becoming thinner as $s \to \pi$, as would be expected.

As N is decreased towards unity numerical inaccuracies become evident in the values of $\mathscr{G}$ as $s \to \pi$; for example the values of $\mathscr{G}$ may be close for s between 0.9π and 0.95π, but there can be significant departures from this trend once $s = 0.99\pi$. Consistent solutions can usually be calculated by reducing the size of the steplength in s, but these also break down closer to $s = \pi$. The reason for the appearance of these errors is associated with the transition region where $F \approx N-1$, since (4.7) implies that

$$F - N + 1 \propto (\pi - s)^{(2-N)/(N-1)} \tag{4.13}$$

along lines where $\bar{\psi}$, and hence $\bar{R}$, is constant. As $N \to 1$ the power of $\pi - s$ in (4.13) becomes large, and the variations of F with s cannot be accurately resolved using a second-order numerical scheme with constant grid spacing in s. The solutions are improved if a stretched coordinate, such as $S = \ln(\pi - s)$, is used near $s = \pi$, but there are still difficulties as $N \to 1$, since the steplength in S must be reduced to order $N-1$. In addition to this problem, the boundary layer thickens more rapidly near the rear stagnation point once N is close to unity. The numerical solutions suggest that most of this thickening is associated with the broadening of the 'plateau' region where $F \approx N-1$, as $N \to 1$. For example, near the value $\bar{\psi}_0$ where $F = N-1$, (4.7) implies that

$$\mathscr{F}^{1/(1-N)} \propto (\pi - s)^{(2-N)/(1-N)} \, (F - N + 1), \tag{4.14}$$

and, assuming $\mathscr{F}^{1/(1-N)}$ is regular at $\bar{\psi} = \bar{\psi}_0$, then

$$\bar{\psi} - \bar{\psi}_0 \propto (\pi - s)^{(2-N)/(1-N)} \, (F - N + 1). \tag{4.15}$$

Therefore the thickness in $\bar{\psi}$ of the region where $|F - N + 1| \leqslant \epsilon$ will increase in proportion to $(\pi - s)^{(2-N)/(1-N)}$ as $s \to \pi$, and this increases rapidly when $N - 1$ is small. This thickening is confirmed qualitatively in figure 5, where F is plotted as a function of $\bar{R}$ for $N = \frac{10}{9}$. The exact power of $\pi - s$ at which the boundary layer thickens is not reproduced in these results, but this is thought to be due to the difficulties with resolution described above.

The rapid thickening of the $F \approx N-1$ 'plateau' region, apparent in figure 5, has a significant displacement effect on the outer layer which is loosely analogous to a separating boundary layer. In this sense Leibovich's (1967*a*) description of the flow

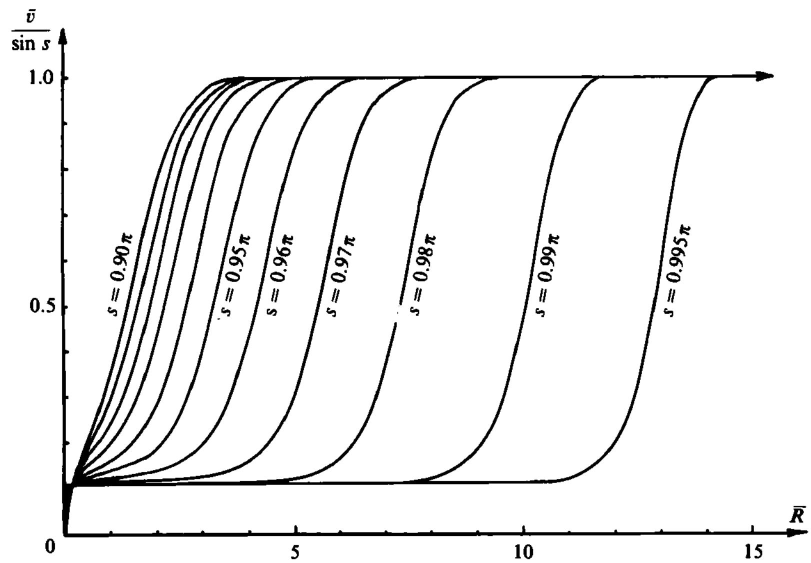

FIGURE 5. Profiles of the scaled $E^{\frac{1}{4}}$ layer velocities $\bar{v}/\bar{v}_e$ close to the rear stagnation point, for $N = \frac{10}{9}$, with a stretched normal coordinate $\bar{R} = \bar{r}\cos\frac{1}{2}s$. The values are given at regular intervals of 0.01π between $s = 0.90\pi$ and 0.99π, with an additional plot at $s = 0.995\pi$.

in this regime as undergoing 'separation without reversed flow' seems to be appropriate. The flow in the plateau region is, to a close approximation, potential stagnation-point flow since the vorticity in this region is only of order $\delta^{(2-N)/(N-1)}$, whereas it is $O(\delta^{-1})$ in both the inner and outer layers. As $N \rightarrow 1$ the size of this region increases, and the velocity scale $N-1$ decreases until it becomes a significant stagnant region when $N = 1$, rather like a separation bubble.

At the outer edge of the boundary layer, where $\bar{\psi} \gg 1$, (4.7) implies that

$$F - 1 \propto (\pi - s)^{N-2}\mathscr{F}(\bar{\psi}), \tag{4.16}$$

and numerical evidence suggests that $\mathscr{F}$ decays at least exponentially in $\bar{\psi}$. Differentiating (4.16) with respect to $\bar{r}$,

$$\frac{\partial \bar{V}}{\partial \bar{r}} \propto (\pi - s)^N \mathscr{F}'(\bar{\psi}), \tag{4.17}$$

and using $\bar{\psi} \approx (\pi - s)\,\bar{r}$ gives that

$$\frac{\partial \bar{V}}{\partial \bar{r}} \propto \bar{r}^{-N}\,\bar{\psi}^N \mathscr{F}'(\bar{\psi}), \tag{4.18}$$

which matches directly onto (3.8). This matching also implies that ζ is $O(\delta^{-1})$ in the wake region, but since $\mathscr{F} \rightarrow 0$ as $\psi \rightarrow \infty$ this region is only of thickness $O(\delta)$ in ψ, or $O(\delta)$ in θ for finite values of r.

The displacement thickness deduced from (4.15), which is proportional to $(\pi - s)^{1/(1-N)}$, increases rapidly near $s = \pi$ once N is close to unity. Therefore the thickness of the wake region will increase as $N \rightarrow 1$, so that once $N = 1$ it is large enough to modify the exterior flow (2.16). Beyond this point boundary-layer separation might be expected and a large stagnant bubble should form behind the cylinder from the remnants of the 'plateau' region.

5. The case $N < 1$

Once $N < 1$ the condition (4.1) is satisfied at a position upstream of the rear stagnation point, and the possibility of flow separation must be entertained. In fact, the trends in the flow as $N \rightarrow 1$, outlined in §4, do seem to indicate that the boundary layer will separate for all $N < 1$. In particular, as N approaches unity from above the outer inviscid layer effectively separates from the wall, although it does remain within the boundary layer, leaving an inviscid irrotational flow beneath it. This irrotational region becomes stagnant at $N = 1$, similar to a separation bubble, although it remains within the boundary layer in the sense that $\overline{\psi}$ is $O(\delta)$ in this region.

When the boundary layer separates from the wall it might be expected that the free shear layers leaving from the wall, on both sides of the cylinder, would enclose a finite region of recirculating flow. However, in a rotating fluid the vorticity in a closed region will always decay to zero, in accordance with (2.15), and therefore the flow within these regions will always satisfy $\nabla^2\psi = 0$. It follows, since ψ is constant on the bounding streamlines, that ψ must be constant within the closed region and therefore the leading-order flow within the separation bubble is stagnant. In a non-rotating fluid Smith (1977) shows that the position of the separated region can be described by Kirchhoff free-streamline theory, in which the velocity along the free streamlines is constant. An analogous procedure can be followed for a rotating fluid, except that velocities along the free streamlines will be linear in the streamwise direction, and the two free shear layers will meet to form a closed separated region (Page 1985). An important feature of these free $E^{\frac{1}{4}}$ layers is that there is no $E^{\frac{1}{3}}$ layer embedded within them, and therefore, like the wake in Page (1983), they are of a different character to the free $E^{\frac{1}{4}}$ layers forced by a velocity discontinuity in Stewartson (1957).

As N is decreased from unity towards zero the size of the separated bubble becomes infinite, since the length of the free shear layers is proportional to N^{-1}, and corresponds to that described in Smith (1977) as $N \rightarrow 0$. This correspondence between the flow for $N \ll 1$ and that in a non-rotating fluid was also noted in Page (1983).

In an experimental situation the flow described above will not usually be attained owing to the development of instabilities in the free shear layers (which will contain an inflection point in the velocity profiles). These instabilities lead to a periodic unsteadiness in the flow, as observed by Boyer (1970) and Boyer & Davies (1982) for some values of $N < 1$, with vortices being shed from the cylinder into the wake. Similar instabilities were observed in free $E^{\frac{1}{4}}$ layers by Hide & Titman (1967), although in their case the free shear layer was not attached to an obstacle, but was forced by a velocity discontinuity on the lid of their container. This instability is described in detail in Busse (1968), Seigmann (1974), Hashimoto (1976) and, more recently, in Niino & Misawa (1984).

6. Conclusion

The analysis in the preceding sections shows the development of boundary-layer separation for the flow past a circular cylinder in a rotating frame as the Rossby number is increased, at a fixed Ekman number. This is achieved by using several results known from an analogous problem in magnetohydrodynamics and then introducing a new form of solution for the boundary-layer flow in the important parameter regime that leads up to separation. The form of this solution gives an indication to the nature of the flow once separation has occurred.

To aid in the interpretation of the results of §§3–5 in a rotating-fluid context, the principal features will be outlined for the various parameter regimes of $\lambda = l/2N$.

(a) $0 \leqslant \lambda \leqslant \frac{1}{4}l$

The boundary-layer flow is fully attached and can be described by a similarity solution in the neighbourhood of the rear stagnation point. There is a weak wake behind the cylinder in which vorticity shed by the boundary decays exponentially. Once $\lambda \geqslant \frac{1}{6}l$ the displacement thickness at the rear stagnation point becomes infinite.

(b) $\frac{1}{4}l < \lambda \leqslant \frac{1}{2}l$

The boundary-layer flow in this regime splits into two layers near the rear stagnation point. The inner 'viscous' layer is governed by a similarity solution related to that in *(a)*, while the outer 'inviscid' layer is dominated by the advection of boundary-layer vorticity from upstream of the stagnation point. The inner layer is characterized by streamfunction values of order $\theta\delta$, where θ is the angle from the rear stagnation point and δ is the scale thickness of the boundary layer, while the stream function is $O(\delta)$ in the outer layer. For values of λ near $\frac{1}{2}l$ these two layers are separated by a region of effectively irrotational flow which becomes larger as $\lambda \rightarrow \frac{1}{2}l$. A wake extends behind the cylinder, transporting the vorticity shed from the outer layer.

(c) $\lambda > \frac{1}{2}l$

The boundary layer separates from the cylinder before the rear stagnation point is reached, and a bubble of stagnant fluid forms behind the cylinder. This bubble is bounded by free $E^{\frac{1}{4}}$ layers, and it distorts the flow around the cylinder significantly. As λ is increased, this bubble grows proportionately until, as $\lambda \rightarrow \infty$, the flow is identical with the equivalent two-dimensional flow in a non-rotating fluid.

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